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# AMPLITUDE EQUATION FOR STATIONARY CONVECTION IN A VISCOELASTIC MAGNETIC FLUID

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We report theoretical and numerical results on convection for a magnetic fluid in a viscoelastic carrier liquid. We focus on the stationary convection for idealized boundary conditions. We obtain explicit expressions of convective thresholds in terms of the control parameters of the system. Close to bifurcation, the coefficients of the corresponding amplitude equation are determined analytically. The effect of the rheology and Kelvin force are emphasized and, finally, the nature of the bifurcation is discussed.

*Keywords*: Thermal convection; magnetic fluid; viscoelastic fluid.

## 1. Introduction

Ferrofluids are magnetic fluids formed by a stable colloidal suspension of magnetic nanoparticles dispersed in a carrier liquid. Without an applied external magnetic field, the orientations of the magnetic moments of the particles are random resulting in a vanishing macroscopic magnetization (magnetic disorder). An external magnetic field, however, easily orients the particles' magnetic moments and a large (induced) magnetization is obtained. There are two main features that distinguish ferrofluids from ordinary fluids, the polarization force and the body couple. In the last decades, much efforts have been dedicated to the study of the phenomenon of convective mechanism in ferrofluids. In addition, heat transfer through magnetic fluids, in particular, have been one of the leading areas of scientific study due to its technological applications [Berkovsky *et al.*, 1973]. The ferrofluid convection has application in high-power capacity transformer system where the ferrofluid is used as a material in the core as well as a coolant in the transform. To activate convective cooling, knowledge of concentration gradient, which will induce convection is required. Besides, an important application of ferrofluids is in biomedicine where the carrier liquid is the blood [Kim *et al.*, 2007]; and it has rheological proprieties. Hence, the detailed study of viscoelastic magnetic fluid is quite important and in order.

The first continuum description of the magnetic fluids was due to [Neuringer & Rosensweig, 1964]. Later, Finlayson [1970] studied the convective instability of a magnetic fluid for a fluid layer heated from below in the presence of a uniform vertical magnetic field. It discussed both shear free and rigid horizontal boundaries using the linear stability method. Gotoh and Yamada [1982] carried out the same study by assuming the fluid to be confined between two magnetic pole pieces. A weakly nonlinear analysis, in a strong external field of this problem was considered by Blennerhassett *et al.* [1991]. Convective instability for a rotating layer of a magnetic fluid has been studied by Gupta and Gupta [1979] and by Venkatasubramanian and Kaloni [1994]. An amplitude for stationary convection with idealized boundary condition was derived in [Kaloni & Lou, 2004a]. The Küppers–Lortz instability for a rotating magnetic fluid was formulated by Auernhammer and Brand [2000]. Ryskin and Pleiner [2004], using the nonequilibrium thermodynamics have derived a complete set of equations to describe ferrofluids in an external magnetic field. This description is made in terms of a binary mixture where the magnetophoretic effect, as well as magnetic stresses, have been taken into account in the static and dynamic parts of the ferrofluid equations. When the magnetophoretic effect is neglected, we have analyzed the thermal convection for rotating ferrofluid for an idealized boundary condition for the typical conductive state in the stationary case and it was found that an analytical expression for the Rayleigh number can be a function of control parameters [Laroze *et al.*, 2006]. Recently, the weakly nonlinear analysis for stationary convection in a rotating magnetic binary mixture was studied in [Laroze *et al.*, 2009]. Other effects such as the buoyancy-surface tension effects, nonuniform thermal gradient, magnetization constitutive equation, etc., have also been studied in [Qin & Kaloni, 1994; Shivakumara *et al.*, 2002; Kaloni & Lou, 2004b, 2005].

On the other hand, the viscoelastic properties of fluids appear in the constitutive equation, which relates the stress and strain rate tensors. Finding this relation, which should generalize the linear dependence characteristic of Newtonian fluids, is the main purpose of rheology. The simplest constitutive equation capable of describing realistically the viscoelastic properties is given by the so-called Oldroyd model [Joseph, 1990]. In this model, the stress tensor is decomposed into both a polymeric contribution and a solvent contribution. The convection on such fluids have been performed by various authors for different cases, either with free-free or rigid-rigid boundary conditions [Vest & Arpaci, 1969; Sokolov & Tanner, 1972; Kolkka & Ierly, 1987; Martinez-Mardones & Perez-Garcia, 1990; Martinez-Mardones *et al.*, 1994; Martinez-Mardones *et al.*, 1995; Martinez-Mardones *et al.*, 1996; Park & Lee, 1995, 1996; Laroze *et al.*, 2005; Laroze *et al.*, 2007a; Laroze *et al.*, 2007b]. It has been found that, besides the usual stationary convection, oscillatory states can also be obtained at onset. Which type of convection — stationary or oscillatory — appears first is determined by the values of the rheological parameters.

In this work, we study a magnetic fluid in a viscoelastic carrier liquid. We introduce the rheologic properties in a constitutive equation for the extra stress tensor. As a first step, we only focus our work on the stationary convection phenomenon for idealized boundary conditions. In particular, the linear stability analysis of the conduction state is performed in the framework of Navier–Stokes equations and we obtain explicit expressions of convective thresholds in terms of the control parameters. The weakly nonlinear analysis for stationary convection is performed and the explicit coefficients of the amplitude equation are calculated and we found that a supercritical–subcritical transition can appear as a consequence of the rheological properties. The paper is organized as follows: In Sec. 2, the basic hydrodynamic equations for viscoelastic magnetic fluid convection are presented. In Sec. 3, the linear stability analysis of the conduction state is performed. In Sec. 4, the weakly nonlinear analysis is developed and the explicit expression for the coefficients of the stationary amplitude equation is found. Finally, conclusions are presented in Sec. 5.

## 2. Theoretical Model

We consider a layer of incompressible magnetic fluid in a viscoelastic carrier liquid, of thickness $d$, parallel to the $xy$-plane, with very large horizontal extension, in a gravitational field $\mathbf{g}$ and submitted to a vertical temperature gradient. The magnetic fluid properties can be modeled as electrically nonconducting superparamagnets, and it is assumed to be placed in a magnetic field $\mathbf{H}$ parallel to $\hat{\mathbf{z}}$, which would be homogeneous if the magnetic fluid were absent. Let us choose the $z$-axis such that $\mathbf{g} = -g\hat{\mathbf{z}}$

and that the layer has its interfaces at $z = -d/2$ and $z = d/2$. A static temperature difference across the layer is imposed, $T(z = -d/2) = T_0 + \triangle T$ and $T(z = d/2) = T_0$. Under the Boussinesq approximation, the balance equations read as

$$\nabla \cdot \mathbf{v} = 0, \tag{1}$$

$$\rho_0 d_t \mathbf{v} = -\nabla p_{\text{eff}} + \nabla \cdot \overline{\boldsymbol{\tau}} + \rho \mathbf{g} + \mathbf{M} \cdot \nabla \mathbf{H}, \tag{2}$$

$$\frac{c_{v,\mathbf{H}}}{T_0} d_t T + \chi_T \mathbf{H}_0 \cdot d_t \mathbf{H} = \overline{\kappa} \nabla^2 T + D_s \chi_c \mathbf{H}_0 \cdot \nabla^2 \mathbf{H}, \tag{3}$$

where $d_t f = \partial_t f + \mathbf{v} \cdot \nabla f$ is the convective derivative and $\mathbf{v}$ being the velocity field, $p_{\text{eff}}$ the effective pressure, $\rho$ the mass density, $\rho_0$ a reference mass density, $\overline{\boldsymbol{\tau}}$ the extra stress tensor, $\mathbf{M}$ the magnetization field, $\mathbf{H}$ the magnetic field, $c_{v,\mathbf{H}}$ the specific heat capacity at constant volume and magnetic field, $T$ the temperature, $T_0$ a reference temperature, $\chi_T$ the pyromagnetic coefficient, $\overline{\kappa}$ the thermal diffusivity.

For the total density we use the following linear state equation

$$\rho = \rho_0 (1 - \alpha_T \triangle T + \alpha_H \mathbf{H}_0 \cdot \triangle \mathbf{H}) \tag{4}$$

where $\alpha_T$ and $\alpha_H$ are the thermal and the magnetic expansion coefficients, respectively; and $\triangle f = f - f_0$. In addition, for the magnetic field $\mathbf{H}$ and the magnetic induction $\mathbf{B}$, we suppose that the system is not a conductor, that is, it is governed by the Maxwell equation:

$$\nabla \times \mathbf{H} = \mathbf{0}, \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{6}$$

Furthermore, the relationship between these fields $\mathbf{B} = \mathbf{H} + \mathbf{M}$ and note that from Eq. (6) that the magnetic field can be in the form of the scalar magnetic potential $\mathbf{H} = -\nabla \phi$. For the magnetization field we use the typical phenomenological state equation $\mathbf{M} = \hat{\mathbf{H}} M(T, H)$:

$$M(T, H) = M_0 - \chi_T \triangle T + \chi_H \mathbf{H}_0 \cdot \triangle \mathbf{H} \tag{7}$$

Moreover, a constitutive equation relating the extra stress tensor $\overline{\boldsymbol{\tau}}$ and the shear rate has to be introduced. For a simple (Newtonian) fluid this constitutive equation is $\overline{\boldsymbol{\tau}} = 2\nu \overline{\mathbf{D}}$ where $\nu$ kinematic viscosity and $\overline{\boldsymbol{\tau}}$ is the symmetric part of the velocity field gradient. However, the rheology of complex fluids can be well described by the Oldroyd model:

$$(1 + \lambda_1 D_t) \overline{\boldsymbol{\tau}} = 2\nu (1 + \lambda_2 D_t) \overline{\mathbf{D}}, \tag{8}$$

in which $\nu$ is the static viscosity, $\lambda_1$ is the relaxation time and $\lambda_2$ is the retardation time, parameters which characterize viscoelasticity. The symbol $D_t$ in Eq. (8) denotes an invariant (frame-indifferent) time derivative, defined as

$$D_t \overline{\boldsymbol{\tau}} = (\partial_t + \mathbf{v} \cdot \nabla) \overline{\boldsymbol{\tau}} + \overline{\boldsymbol{\tau}} \cdot \overline{\mathbf{W}} - \overline{\mathbf{W}} \cdot \overline{\boldsymbol{\tau}} + a(\overline{\boldsymbol{\tau}} \cdot \overline{\mathbf{D}} + \overline{\mathbf{D}} \cdot \overline{\boldsymbol{\tau}}), \tag{9}$$

where $\overline{\mathbf{W}}$ is the skew-symmetric part of the velocity field gradient; also $a$ is a phenomenological parameter in the range $-1$ to $+1$. For $a = -1$ the lower convective Jeffreys model (Oldroyd B) is obtained, $a = 0$ leads to the so-called corotational Jeffreys model, and $a = 1$ to the upper convective Jeffreys model (Oldroyd A). Let us comment that the coefficient $a$ is not completely free in choosing its value [Pleiner *et al.*, 2004] and that when $\lambda_2 = 0$ gives a Maxwellian fluid, while a Newtonian fluid requires $\lambda_1 = 0$ and $\lambda_2 = 0$.

Now we analyzed the boundary conditions (BCs); thermal BCs, a static temperature difference across the layer is imposed, $T(z = -d/2) = T_0 + \triangle T$ and $T(z = d/2) = T_0$; for the magnetic BCs we used the typical continuity condition $\mathbf{n} \times (\mathbf{H}_{in} - \mathbf{H}_{ex}) = \mathbf{0}$ and $\mathbf{n} \cdot (\mathbf{B}_{in} - \mathbf{B}_{ex}) = 0$, where $\mathbf{n}$ is a unitary vector normal to the boundaries. Moreover, for velocity we will consider unrealistic free interfaces. Hence, from Eqs. (1)–(9) and using these boundary conditions, the conductive basic state is

$$\mathbf{v}_{\text{con}} = \mathbf{0}, \tag{10}$$

$$T_{\text{con}}(z) = \overline{T} - \beta z, \tag{11}$$

$$\mathbf{H}_{\text{con}}(z) = \mathbf{H}_0 (1 + \lambda \beta z), \tag{12}$$

where $\beta = (\triangle T/d)$ and $\lambda = \chi_T/(1 + \chi_H)$. Therefore, the perturbation equation of the conducting

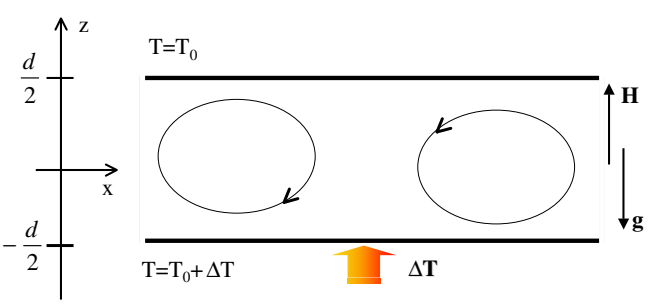



Fig. 1. A vertical cut through the fluid layer. Note the $y$-axis point into the $xz$-plane.

state can be written as

$$\nabla \cdot \mathbf{v} = \mathbf{0} \tag{13}$$

$$(\partial_t + \mathbf{v}\cdot\nabla)(\theta - M_4\partial_z\phi) = (1-M_4)w + \nabla^2\theta \tag{14}$$

$$P^{-1}(\partial_t + \mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p_{\text{eff}} + \nabla\cdot\tau + R_a\Pi_1\hat{\mathbf{z}} + R_aM_1\theta\nabla(\partial_z\phi) \tag{15}$$

$$(1+\Gamma D_t)\overline{\boldsymbol{\tau}} = (1+\Gamma\Lambda D_t)\overline{\boldsymbol{D}} \tag{16}$$

$$(\partial_{zz} + M_3\nabla_\perp^2)\phi - \partial_z\theta = 0 \tag{17}$$

$$\nabla^2\phi_{\text{ext}} = 0 \tag{18}$$

where $\Pi_1(\theta,\phi) = (1+M_1)\theta - (M_1 - M_5)\partial_z\phi$. Also the following groups of dimensionless numbers have been introduced: (a) (pure fluids) the Rayleigh number $R_a = \alpha_T g \triangle T d^3/\kappa\nu$ accounting for buoyancy effects and the Prandtl number $P = \nu/\kappa$, relating viscous and thermal effects; (b) (magnetic fluid) the strength of magnetic force relative to buoyancy $M_1 = \beta\chi_T^2 H_0^2/(\rho_0 g\alpha_T(1+\chi)))$, the nonlinearity of magnetization $M_3 = 1 - (\chi_H H_0^2)/(1+\chi)$ is a measure of the deviation of the magnetization curve from the linear behavior $M_0 = \chi_H H_0$, the relative strength of temperature dependence of the magnetic susceptibility $M_4 = \chi_T^2 H_0^2 T_0/c_{v,\mathbf{H}}(1+\chi)$ and the ratio of magnetic to thermal buoyancy $M_5 = \alpha_H\chi_T H_0^2/(\alpha_T(1+\chi))$; (c) (viscoelastic fluid) the Deborah number $\Gamma = \lambda_1\overline{\kappa}/d^2$ and the ratio between retardation and stress relaxation times $\Lambda = \lambda_2/\lambda_1$. Note that, for $\Gamma = 0$ one recovers a Newtonian fluid and for $\Lambda = 0$ a Maxwellian fluid is obtained.

Let us comment about the numerical values of the parameters; the parameter $R_a$ may be changed in several orders of magnitude. The magnetic numbers have the following order of magnitude $M_1 = 10^{-4} - 10^1$, $M_3 \approx 1.1$, $M_4 \approx M_5 \approx 10^{-6}$ [Laroze *et al.*, 2006]. The values of $M_4$, $M_5$ are very small, so in the following analysis, we are not going to take them into account.

For the sake of simplicity, the analysis was limited to two-dimensional motion. Therefore, we can use a stream function $\psi(x,z,t)$ that gives the velocity field $(v_x, v_z)$. Due to its symmetry properties, the extra stress tensor has only three independent coefficients: $\tau_{xx}, \tau_{zz}, \tau_{zx} = \tau_{xz}$. Instead of these individual components, the following three scalar quantities are usually considered in rheology: the trace $U = \tau_{xx} + \tau_{zz}$, the normal stress difference $S = \tau_{xx} - \tau_{zz}$ and the in-plane shear stress $\tau_{zz}$. Using the afore mentioned assumptions, we can write Eqs. (13)–(17) in the following way:

$$\pounds\mathbf{u} + \mathbf{N}(\mathbf{u}\,|\,\mathbf{u}) = 0 \tag{19}$$

with $\mathbf{u} = (\psi,\theta,\phi,\tau,S,U)^T$ and where $\pounds$ is the spatio-temporal linear operator and $\mathbf{N}$ stands for the nonlinear part of the respective equations, and they are given by:

$$\pounds = \begin{pmatrix} \partial_t\nabla^2 & PR_a(1+M_1)\partial_x & -PR_aM_1\partial_{zz}^2 & -P\nabla^2 & -P\partial_{xz}^2 & 0 \\ \partial_x & \partial_t - \nabla^2 & 0 & 0 & 0 & 0 \\ 0 & \partial_z & -(\partial_{zz}^2 + M_3\partial_{xx}^2) & 0 & 0 & 0 \\ -(\Gamma\Lambda\partial_t + 1)\nabla^2 & 0 & 0 & 1+\partial_t\Gamma & 0 & 0 \\ -4(\Gamma\Lambda\partial_t\partial_{xx}^2 + \partial_{xz}^2) & 0 & 0 & 0 & 1+\partial_t\Gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\partial_t\Gamma \end{pmatrix} \tag{20}$$

and

$$\mathbf{N} = \begin{pmatrix} -J(\psi,\nabla^2\psi) + PR_aM_1\partial_z\theta\nabla^2\phi \\ -J(\psi,\theta) \\ 0 \\ \Gamma\Lambda[J(\psi,\triangle^2\psi) + 2(\partial_{xz}^2\psi)\nabla^2\psi] - \Gamma\left[J(\psi,\tau) + \frac{1}{2}(\nabla^2\psi)S - \frac{1}{2}\mathbf{a}(\partial_{xz}^2\psi)U\right] \\ \Gamma\Lambda[4J(\psi,\partial_{xz}^2\psi) - 2(\nabla^2\psi)(\triangle^2\psi)] + \Gamma[-J(\psi,S) + 2(\nabla^2\psi)\tau + 2\mathbf{a}(\partial_{xz}^2\psi)S] \\ 2\Gamma\Lambda\mathbf{a}[4(\partial_{xz}^2\psi)(\partial_{xz}^2\psi) + (\triangle^2\psi)(\nabla^2\psi)] - \Gamma[-J(\psi,U) + 2\mathbf{a}(\nabla^2\psi)\tau + 2\mathbf{a}(\partial_{xz}^2\psi)S]c \end{pmatrix} \tag{21}$$

where $J(f,g) = \partial_x f \partial_z g - \partial_z f \partial_x g$ and $\triangle^2 = \partial^2_{zz} - \partial^2_{xx}$. In the next sections, we will develop the linear and weakly nonlinear analysis for stationary convection.

## 3. Linear Stability Analysis

In order to calculate the linear stability analysis, we only need the linear parts of Eq. (19). The space-temporal dependencies of $\mathbf{u}(\mathbf{r},t)$ are separated by the usual normal mode expansion:

$$\mathbf{u}(\mathbf{r},t) = \tilde{\mathbf{u}}(z)\exp[i\,kx + st], \tag{22}$$

with $\tilde{\mathbf{u}} = (\tilde{\psi},\tilde{\theta},\tilde{\phi},\tilde{\tau},\tilde{S},\tilde{U})^T$, where $k$ is the horizontal wavenumber and where $s = \sigma + i\Omega$ denotes the complex eigenvalues; $\sigma$ is the growth factor of a perturbation, and $\Omega$ its frequency. With the purpose of finding analytical expressions we impose the following boundary conditions $\tilde{\psi} = D^2\tilde{\psi} = \tilde{\theta} = D\tilde{\phi} = \tilde{\tau} = D\tilde{S} = \tilde{U} = 0$ in $z = \pm 1/2$; with $D^n f = \partial^n_z f$. In the following, we will only focus on the stationary convection case. Hence, the eigenvalue problem gives the Rayleigh number and the stationary bifurcation is obtained for the condition of marginal linear stability $s = 0$. For the fundamental mode, we can introduce the anszat on the stream function:

$$\tilde{\psi}(z) = \mathrm{Cos}[\pi z]. \tag{23}$$

Therefore, $\pounds\mathbf{u} = 0$ leads

$$\tilde{\theta}(z) = -\frac{k}{q^2}\mathrm{Cos}[\pi z], \tag{24}$$

$$\tilde{\phi}(z) = \frac{\pi}{q^2(k^2M_3 + \pi^2)}\mathrm{Sin}[\pi z], \tag{25}$$

$$\tilde{\tau}(z) = -\Sigma\mathrm{Cos}[\pi z], \tag{26}$$

$$\tilde{S}(z) = 4\pi k\mathrm{Sin}[\pi z], \tag{27}$$

$$\tilde{U}(z) = 0, \tag{28}$$

where $q^2 = k^2 + \pi^2$ and $\Sigma = \pi^2 - q^2$. As solvability condition to the linear part of Eq. (19) we find the following relation of the parameters to have a stationary instability (marginal curve)

$$R_a = \frac{(k^2M_3 + \pi^2)(\pi^2 + k^2)^3}{k^2[\pi^2(1 + M_1)(\pi^2 + k^2M_3) - M_1\pi^2]} \tag{29}$$

Note that, Eq. (29) is the same as $R_a$ for simple ferrofluid [Finlayson, 1970]; hence the viscoelastic effects do not show up here. The minimum of the marginal curve ($\partial_k R_a = 0$) gives the critical wavenumber $k_c$ and, the critical Rayleigh number, $R_{ac}$, can be obtained by replacing it in formula (29).

## 4. Weakly Nonlinear Analysis

In this section, the evolution of the pattern is developed by means of a multiple scale analysis used by Newell, Whitehead and Segel [Newell & Whitehead, 1969; Segel, 1969]. We will assume that, a small amplitude convection cell is imposed on the basic flow. For values of the control parameter $R_a$ given by Eq. (29) close to its threshold value $R_{ac}$, the bifurcation parameter will be

$$\epsilon^2 = \frac{R_a - R_{ac}}{R_{ac}} \tag{30}$$

We expand all functions in $\epsilon$ and assume that all variations of the linearized solutions can be incorporated into an amplitude function $A$. If this amplitude is of the size of $\epsilon(O(\epsilon))$ then the interaction of the cell with itself forces a second harmonic and a mean state of correction of size $O(\epsilon^2)$ and these in turn drive an $O(\epsilon^3)$ correction to the fundamental component of the imposed roll. A solvability criterion for this last correction yields an equation of complex amplitude $A(X,T)$ of the imposed disturbance, the Ginzburg–Landau (GL) equation. To simplify the problem we assume the formation of cylindrical rolls with axis parallel to the $y$-axis so that the $y$-dependence disappears from (19). The $z$-dependence is contained entirely in the sine and cosine functions, which ensures that the free-free boundary conditions are satisfied. Therefore, our expansion is

$$\mathbf{u} \to \epsilon A\{\mathbf{u}^{(0)} + \epsilon\mathbf{u}^{(1)} + \epsilon^2\mathbf{u}^{(2)} + \Theta(\epsilon^3)\} \tag{31}$$

Here $\mathbf{u}^{(0)}$ is the linear solution. Inserting the above equation into governing Eq. (19), we expand after which we obtain a hierarchy of equations

$$\epsilon^1: \quad \pounds^{(0)}\mathbf{u}^{(0)} = 0 \tag{32}$$

$$\epsilon^2: \quad \pounds^{(0)}\mathbf{u}^{(1)} = -N(\mathbf{u}^{(0)}\,|\,\mathbf{u}^{(0)}) \tag{33}$$

$$\epsilon^3: \quad \pounds^{(0)}\mathbf{u}^{(2)} = -\pounds^{(2)}\mathbf{u}^{(0)} - N(\mathbf{u}^{(0)}\,|\,\mathbf{u}^{(1)}) - N(\mathbf{u}^{(1)}\,|\,\mathbf{u}^{(0)}) \tag{34}$$

These equations must be solved iteratively each time fulfilling the solvability condition

$$\langle\mathbf{u}^\dagger\,|\,\mathrm{r.h.s.}\rangle = \mathbf{0} \tag{35}$$

where $\mathbf{u}^\dagger$ are the solution of the kernel of the linear adjoint problem ($\pounds^+\mathbf{u}^\dagger = 0$). Also, r.h.s means the corresponding right-hand side of the perturbation equations and $\langle\circ\rangle$ denote the inner product which is an average over a suitable volume. The solvability

condition at $O(\epsilon^3)$ leads to an equation for the amplitude $A$ of the type [Cross & Hohenberg, 1993]:

$$\tau_0 \frac{\partial A}{\partial t} = \xi_0^2 \frac{\partial^2}{\partial x^2} A + \epsilon^2 A - g|A|^2 A \tag{36}$$

Equation (36) is the GL equation and describes the variation of the slow time scale $\epsilon^2 t$ and slow spatial scale $\epsilon x$, perpendicular to the rolls. For $\epsilon = 0$, we do not obtain the GL equation. The coefficients of this equation can be calculated as [Auernhammer & Brand, 2000]:

$$\tau_0 = \left( R_{ac} \frac{\partial \sigma}{\partial R_a} \right)^{-1}_{R_{ac}} \tag{37}$$

$$\xi_0^2 = \frac{1}{2R_{ac}} \left( \frac{\partial^2 R_a}{\partial k^2} \right)_{R_{ac}} \tag{38}$$

$$g = \tau_0 \langle \mathbf{u}^\dagger \,|\, N(\mathbf{u}^{(0)} \,|\, \mathbf{u}^{(1)}) + N(\mathbf{u}^{(1)} \,|\, \mathbf{u}^{(0)}) \rangle_{R_{ac}} \tag{39}$$

The coefficients $\tau_0$ and $\xi_0^2$ are known as growth rate amplitude and the curvature of the marginal stability curve, respectively. The coefficient $g$ is known as the nonlinear cubic coefficient and for $g > 0$ we get the forward bifurcation (supercritical bifurcation), for $g < 0$ we get the backward bifurcation (subcritical bifurcation), and at $g = 0$ we get the tricritical bifurcation point which is the transition point between subcritical and supercritical bifurcations; and this critical transition is also denominated as the supercritical quintic bifurcation. After straight forward calculations, the explicit expressions of these coefficients can be written as

$$\tau_0 = \frac{1 + P^{-1}}{q_c^2} - \Xi \tag{40}$$

$$\xi_0^2 = \frac{2}{q_c^2} - \frac{M_1 M_3 \pi^2}{(M_3 k_c^2 + \pi^2)((1 + M_1) M_3 k_c^2 + \pi^2)} \tag{41}$$

$$g = \frac{\pi^2 q_c^2 (M_3(1 + 2M_1)k_c^2 + \pi^2)}{2(q_c^2(1 + M_1)M_3 + \pi^2)} + \frac{a^2 \Xi \Gamma}{q_c^4}(9q_c^{16} + 8\pi^2 k_c^2 \Sigma_c^4 + 144\pi^4 k_c^4) - \frac{\Xi\Gamma}{q_c^4}(9q_c^8 \Sigma_c^4 + 4\pi^2 k_c^2 q_c^8 + 64\pi^4 k_c^4), \tag{42}$$

where $\Xi = \Gamma(1 - \Lambda)$. Note that, we can recover some previous results, for instance, if we take $\Gamma = 0$, we obtain the exact coefficients for a magnetic Newtonian fluid found in [Kaloni & Lou, 2004a]. In addition, imposing $M_1 = M_3 \equiv 0$, Eqs. (40)–(42) are reduced to the coefficients for viscoelastic fluid found in [Martinez-Mardones *et al.*, 2003].

Figure 2 shows the normalized cubic nonlinear coefficient $g/g_{\max}$ as a function of $M_1$ for different values of the phenomenological parameter $a$ and we observe that $g$ increases when $M_1$ increases and that $g$ decreases when $a$ decreases. Also, the numerical values of $g$ as a function of $M_1$ in the Newtonian case are bigger than the viscoelastic ones, as shown in the inset.

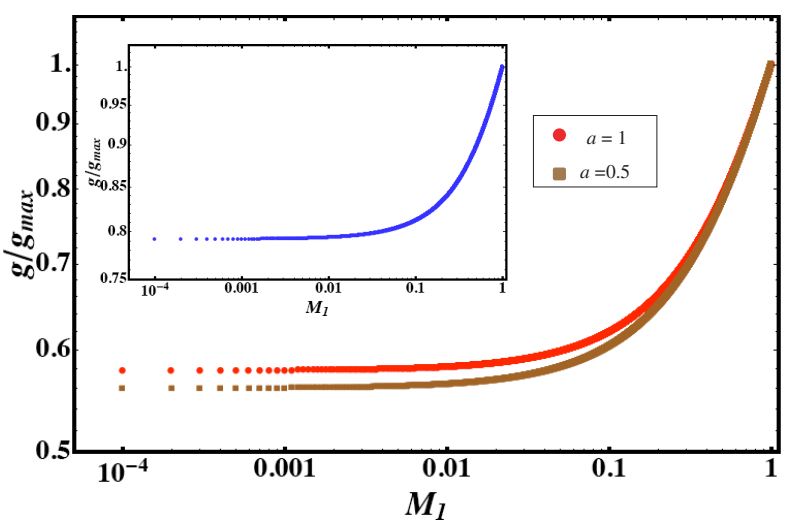



Fig. 2. Normalized cubic nonlinear coefficient $g/g_{\max}$ as a function of $M_1$ at $\Lambda = 0.75$, $\Gamma = 0.2$ and $M_3 = 1.1$ with $a = 1$ and $a = 0.5$. The inset shows $g/g_{\max}$ as a function of $M_1$ for Newtonian fluid ($\Lambda = 1$) at $M_3 = 1.1$.

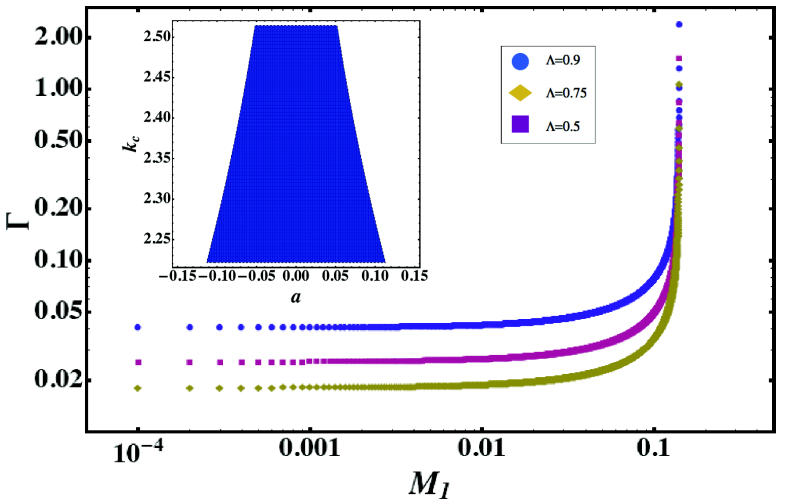



Fig. 3. Phase diagram of $\Gamma - M_1$ for the supercritical–subcritical transition at $M_3 = 1.1$ and $a = 0.1$ with three different values of $\Lambda$. The inset shows the range of validity of the transition at $M_3 = 1.1$ for $M_1 \in (10^{-4}, 1)$.

On the other hand, from Eq. (42) we find that $g = 0$ satisfies the following relationship:

$$\Gamma^2_{g=0} = \frac{\pi^2(k_c^2(1+2M_1)M_3 + \pi^2)q_c^6}{2(\pi^2 + (1+M_1)M_3 q_c^2)(\Lambda - 1)(a^2\eta_c - 64k_c^2\pi^4 - 4k_c^2\pi^2 q_c^8 - 9q_c^8\Sigma_c^4)}, \tag{43}$$

where $\eta = 144k^2\pi^4 + 9q^{16} + 8k^2\pi\Sigma^4$. Hence, a supercritical–subcritical transition appears as a consequence of viscoelastic properties.

Figure 3 shows $\Gamma_{g=0}$ as a function of $M_1$ for the range of parameters where the transition phenomenon occurs. Notice that, for $\Gamma \geq 0$ and $0 \leq \Lambda \leq 1$, Eq. (43) has the following restrictions: $a^2\eta_c - 64k_c^2\pi^4 - 4k_c^2\pi^2 q_c^8 - 9q_c^8\Sigma_c^4 < 0$ and $\Lambda \neq 1$. These restrictions implies that the transition occurs for models like corotational Jeffrey's model because $a$ is near to zero, as shown in the inset of Fig. 3. Also, we remark that our analysis is valid only for explaining the phenomenon when $g > 0$, but we have other ranges of physical parameters as $g < 0$ and in this case, it is necessary to use the Cubic–Quintic GL equation.

## 5. Final Remarks

In the present work, the Rayleigh–Benard convection in a magnetic fluid in a carrier viscoelastic liquid is studied. We determine the stability thresholds for stationary convection with free boundary conditions, giving the same results as obtained in [Finlayson, 1970]. The weakly nonlinear analysis was performed and the analytical expression of the coefficient of the corresponding amplitude equation was obtained. We found that, the viscoelastic effect appears in the growth rate amplitude and for the nonlinear coefficient. Due to the rheological properties, a subcritical–supercritical transition can be produced. Let us comment that, very often the ferrofluids show a finite separation ratio and magnetic separation ratio requiring a binary mixture description; however, for materials where the separation ratio and magnetic separation ratio are not too large, the simple fluid approximation can be used [Ryskin & Pleiner, 2004]; and the present work is based on this feature. Finally, we remark that the oscillatory convention for this system and the inclusion of the binary mixture effects are still in progress and the results will be presented in future works.

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